Problem 2.1 We can write the transformation as perturbation series

$$x' = x + vt + \delta x^{(2)}$$
$$t' = t + \delta t^{(1)} + \delta t^{(2)}$$

The perturbation terms can be obtained via order-by-order comparison

$$t'^2 - x'^2 = t^2 - x^2$$
$$(2t\delta t^{(1)} - 2tvx) + (2t\delta t^{(2)} + [\delta t^{(1)}]^2 - 2x\delta x^{(2)} - v^2t^2) = 0$$

which gives

$$\delta t^{(1)} = vx$$
$$2t(\delta t^{(2)} - \frac{v^2 t}{2}) + 2x(\frac{v^2 x}{2} - \delta x^{(2)}) = 0$$
$$\begin{cases} \delta t^{(2)} = \frac{v^2 t}{2} \\ \delta x^{(2)} = \frac{v^2 x}{2} \end{cases} \quad (x, t \text{ are arbitrary})$$

If we expand the full transformations, we have

$$x' = (x + vt)(1 + \frac{v^2}{2} + O(v^4))$$
$$= x + vt + \frac{v^2 x}{2} + O(v^3)$$
$$t' = t + vx + \frac{v^2 t}{2} + O(v^3)$$

which agree with the results obtained from perturbation.

Problem 2.2

$$E = 7\,\text{TeV}$$
$$\gamma = \frac{E}{m} \approx 7000$$
$$\beta \approx 0.9999999898$$
$$v - c = (1 - \beta)c \approx 3\,\text{ms}^{-1}$$

For the relative velocity we have

$$v_{rel} = \frac{2\beta}{1 + \beta^2} c = c$$

Problem 2.3 The energy of the CMB photon is given by

$$E_\gamma \approx kT_{CMB} = 0.00023\,\text{eV}$$

At threshold, the final products should be at rest in the center of momentum frame. By momentum conservation we have

$$\begin{aligned}
P_p + P_\gamma &= P'_p + P'_\pi \\
m_p^2 + 2E_\gamma(E_p + p_p) &= (m_p + m_\pi)^2 \\
E_p + \sqrt{E_p^2 - m_p^2} &= \frac{(m_p + m_\pi)^2 - m_p^2}{2E_\gamma} = A \\
E_p^2 - m_p^2 &= A^2 - 2AE_p + E_p^2 \\
E_p &= \frac{A^2 + m_p^2}{2A} \\
&\approx 3.1 \times 10^{20}\,\text{eV}
\end{aligned}$$

To find the energy of the outgoing proton, we first find its speed via

$$\begin{aligned}
p_p + E_\gamma &= \gamma\beta(m_p + m_\pi) \\
\gamma\beta &= \frac{E_\gamma + \sqrt{E_p^2 - m_p^2}}{m_p + m_\pi} = B \\
\gamma &= \sqrt{B^2 + 1} \\
&\approx 2.6 \times 10^{11}
\end{aligned}$$

Thus

$$E'_p = \gamma m_p \approx 2.6 \times 10^{20}\,\text{eV}$$

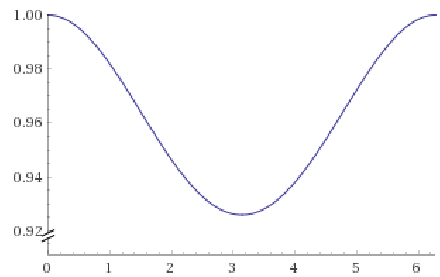
Problem 2.4 Yes, rotate about y-axis by π and then apply P.

Problem 2.5 Typical X-ray energy is on the order of 1 keV which is much larger than the ionization energy of the electron on the order of 10 eV. Therefore for most cases we can neglect the binding energy of the electron.

We can find the frequency of the reflected X-ray via momentum conservation

$$\begin{aligned}
P_\gamma + P_e &= P'_\gamma + P'_e \\
(P'_e)^2 &= (P_\gamma + P_e - P'_\gamma)^2 \\
m_e^2 &= m_e^2 + 2P_e \cdot P_\gamma - 2P_e \cdot P'_\gamma - 2P_\gamma \cdot P'_\gamma \\
0 &= m_e E_\gamma - m_e E'_\gamma - E_\gamma E'_\gamma(1 - \cos\theta) \\
E'_\gamma &= \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e}(1 - \cos\theta)}
\end{aligned}$$

which looks like the following when plotted



If the electron mass goes to zero, the only possible solution becomes $E'_\gamma = E_\gamma$ and $\theta = 0$ because the photon cannot have zero energy. In other words the interaction between the photon and the electron will be turned off in this limit.

In classical EM the frequency of the outgoing radiation produced by the electron is the same as the driving frequency which is the frequency of the incoming photon. Therefore the frequency distribution is a constant function.

Problem 2.6

$$\begin{aligned}\int_{-\infty}^{\infty} dk^0\, \delta((k^0)^2 - (\boldsymbol{k}^2 + m^2))\theta(k^0) &= \int_{-\infty}^{\infty} dk^0 \left(\frac{\delta(k^0 - \omega_k)}{2\omega_k} + \frac{\delta(k^0 + \omega_k)}{2\omega_k}\right)\theta(k^0) \\ &= \int_0^{\infty} dk^0\, \frac{\delta(k^0 - \omega_k)}{2\omega} \\ &= \frac{1}{2\omega_k}\end{aligned}$$

where $\omega_k = \sqrt{\boldsymbol{k}^2 + m^2}$.

The Jacobian for Lorentz tranformation is $J = |\det \Lambda| = 1$. Therefore the measure d^4k is Lorentz invariant.

We can consider the following integral

$$I = \int_{-\infty}^{\infty} d^4k\, \delta(k^2 - m^2)\, \theta(k^0)$$

Both the measure and the integrand are manifestly Lorentz invariant, thus I itself must be Lorentz invariant. We can write I in another form

$$\begin{aligned}I &= \int d^3k \int dk^0\, \delta(k^2 - m^2)\, \theta(k^0) \\ &= \int \frac{d^3k}{2\omega_k}\end{aligned}$$

which gives us the desired result.

Problem 2.7

$$\begin{aligned}
\partial_z(e^{-za^\dagger}ae^{za^\dagger}) &= -a^\dagger(e^{-za^\dagger}ae^{za^\dagger}) + (e^{-za^\dagger}ae^{za^\dagger})a^\dagger \\
&= e^{-za^\dagger}(aa^\dagger - a^\dagger a)e^{za^\dagger} \\
&= 1 \\
e^{-za^\dagger}ae^{za^\dagger} &= z + a \\
ae^{za^\dagger} &= ze^{za^\dagger} + e^{za^\dagger}a
\end{aligned}$$

Thus

$$\begin{aligned}
a|z\rangle &= ae^{za^\dagger}|0\rangle \\
&= z|z\rangle + e^{za^\dagger}a|0\rangle \\
&= z|z\rangle
\end{aligned}$$

$$\begin{aligned}
\langle n|\hat{n}|z\rangle &= \langle n|a^\dagger a|z\rangle \\
n\langle n|z\rangle &= z\sqrt{n}\langle n-1|z\rangle \\
c_n &= \frac{z}{\sqrt{n}}c_{n-1} \\
c_n &= \frac{z^n}{\sqrt{n!}}c_0 \\
&= \frac{z^n}{\sqrt{n!}}
\end{aligned}$$

where $c_n = \langle n|z\rangle$, and $c_0 = \langle 0|e^{za^\dagger}|0\rangle = \langle 0|(1 + za^\dagger + ...)|0\rangle = \langle 0|0\rangle = 1$.
To calculate the uncertainties, we first evaluate the following quantities

$$\begin{aligned}
\langle z|z\rangle &= \sum |c_n|^2 = \sum \frac{|z|^{2n}}{n!} = e^{|z|^2} \\
\langle z|a|z\rangle &= \sum c_n c_{n-1}^* \sqrt{n} = z\sum \frac{|z|^{2(n-1)}}{(n-1)!} = ze^{|z|^2} \\
\langle z|\hat{n}|z\rangle &= \sum c_n c_n^* n = |z|^2 e^{|z|^2} \\
\langle z|aa^\dagger|z\rangle &= \sum c_n c_n^*(n+1) = (|z|^2+1)e^{|z|^2} \\
\langle z|a^2|z\rangle &= \sum c_n c_{n-2}^* \sqrt{n(n-1)} = z^2 e^{|z|^2}
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta q^2 = \langle q^2\rangle - \langle q\rangle^2 &= \frac{z^2 + 2|z|^2 + 1 + (z^*)^2 - (z+z^*)^2}{2m\omega} = \frac{1}{2m\omega} \\
\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 &= \frac{-m\omega}{2}[z^2 - 2|z|^2 - 1 + (z^*)^2 - (z-z^*)^2] = \frac{m\omega}{2}
\end{aligned}$$

Thus

$$\Delta p \Delta q = \frac{1}{2}$$

Assume $|w\rangle = \sum b_n |n\rangle$ is an eigenstate of $a^\dagger$, then we have

$$\begin{aligned}
a^\dagger |w\rangle &= \sum b_n \sqrt{n+1} |n+1\rangle \\
\sum w b_n |n\rangle &= \sum b_{n-1} \sqrt{n} |n\rangle \\
b_n &= b_{n-1} \frac{\sqrt{n}}{w} \\
&= b_0 \frac{\sqrt{n!}}{w^n} \\
&= 0
\end{aligned}$$

In the last line we used $b_0 = 0$. This is because $0 = \langle 0 | a^\dagger | w \rangle = w b_0$. Since all the coefficients vanish, $a^\dagger$ does not have any eigenstate.

Problem 3.1 Each additional derivative simply produces a minus sign after integration by parts, therefore it is easy to see that the equation of motion is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} + etc. = 0$$

Problem 3.2 We can write the infinitesimal Lorentz transform as

$$\Lambda_{\mu\nu} = \delta_{\mu\nu} + \frac{1}{2}\mathcal{J}^{(\alpha\beta)}_{\mu\nu}\omega_{\alpha\beta}$$

where $\omega_{\alpha\beta}$ is the rotation or boost parameter in the $\alpha - \beta$ plane. The factor of $1/2$ is to avoid double-counting (e.g. $1-2$ and $2-1$ are the same rotation).

For example, for an infinitesimal rotation in the $1-2$ plane we have

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\omega_{12} & 0 \\ 0 & \omega_{12} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^\mu_\nu + \omega_{12}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{J}^{(12)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \delta^1_\mu \delta^2_\nu - \delta^1_\nu \delta^2_\mu$$

There are $C^4_2 = 6$ such generators (3 rotations and 3 boosts), and in general we can write

$$\mathcal{J}^{(\alpha\beta)}_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu$$

Equipped with this formula, we can now calculate the Noether current corresponding to Lorentz symmetry.

$$\begin{aligned}
\partial_\mu \mathcal{L}\delta x^\mu &= \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\lambda \phi_n \delta x^\lambda \Big) \\
\partial^\mu \mathcal{L}\mathcal{J}^{(\alpha\beta)}_{\mu\nu} x^\nu &= \partial^\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_n)} \partial^\lambda \phi_n \mathcal{J}^{(\alpha\beta)}_{\lambda\nu} x^\nu \Big) \\
\partial^\mu \big(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_n)} \partial^\lambda \phi_n \mathcal{J}^{(\alpha\beta)}_{\lambda\nu} x^\nu - \mathcal{L}\mathcal{J}^{(\alpha\beta)}_{\mu\nu} x^\nu \big) &= -\mathcal{L}\mathcal{J}^{(\alpha\beta)}_{\mu\nu}\delta^{\mu\nu} \\
\partial^\mu (T^\lambda_\mu \mathcal{J}^{(\alpha\beta)}_{\lambda\nu} x^\nu) &= 0 \\
\partial^\mu (T^\alpha_\mu x^\beta - T^\beta_\mu x^\alpha) &= 0 \\
K_{\alpha\beta\mu} &= T_{\mu\alpha} x_\beta - T_{\mu\beta} x_\alpha
\end{aligned}$$

The third line we used the fact that an antisymmetric tensor contracted with a symmetry tensor gives zero.

Now we want to evaluate this current for a free massive scalar theory. We first calculate the stress-momentum tensor.

$$T_{\mu\nu} = -\partial_\mu \phi \partial_\nu \phi + g_{\mu\nu}\frac{1}{2}(\partial_\lambda \phi)^2 - g_{\mu\nu}\frac{1}{2}m^2\phi^2 = T_{\nu\mu}$$

and also

$$\begin{aligned}\partial^\mu T_{\mu\nu} &= -\partial^2\phi\partial_\nu\phi - \partial^\mu\phi\partial_\mu\partial_\nu\phi + \frac{1}{2}\partial_\nu(\partial_\lambda\phi)^2 - \frac{1}{2}\partial_\nu(m^2\phi^2)\\ &= -\partial^2\phi\partial_\nu\phi - m^2\phi\partial_\nu\phi - \partial^\mu\phi\partial_\mu\partial_\nu\phi + \partial^\lambda\phi\partial_\nu\partial_\lambda\phi\\ &= 0\end{aligned}$$

and

$$K_{\alpha\beta\mu} = x_\beta(\partial_\mu\phi\partial_\alpha\phi + g_{\mu\alpha}(\partial_\lambda\phi)^2 - g_{\mu\alpha}m^2\phi^2) - x_\alpha(\partial_\mu\phi\partial_\beta\phi + g_{\mu\beta}(\partial_\lambda\phi)^2 - g_{\mu\beta}m^2\phi^2)$$

To check that the current satisfies the continuity equation, we take the derivative.

$$\begin{aligned}\partial_\mu K_{\alpha\beta\mu} &= T_{\mu\alpha}\delta^\mu_\beta - T_{\mu\beta}\delta^\mu_\alpha\\ &= T_{\beta\alpha} - T_{\alpha\beta}\\ &= 0\end{aligned}$$

Next we want to calculate the charge corresponding to the boost.

$$\begin{aligned}K_{0i0} &= T_{00}x_i - T_{0i}x_0\\ &= \mathcal{E}x_i - \mathcal{P}_i t\\ Q_i &= \int \mathcal{E}x_i\, d^3x - P_i t\\ &= E\bar{x}_i - P_i t\\ \bar{x}_i &= \frac{Q_i}{E} + \frac{P_i}{E}t\end{aligned}$$

Thus physically the conservation law means that the center of energy of the system $\bar{x}_i$ moves in a straight line.

The final part of the problem follows trivially from the Heisenberg equation.

$$\begin{aligned}\frac{dQ_i}{dt} &= i[H, Q_i] + \frac{\partial Q_i}{\partial t}\\ i\frac{\partial Q_i}{\partial t} &= [H, Q_i]\end{aligned}$$

In general, since Q_i is a linear combination of position and momentum, $[Q_i, H] \neq 0$. Therefore Q_i is not an invariant of the equation of motion.

Problem 3.3 By adding a total derivative the energy-momentum tensor changes by

$$\begin{aligned}\delta T_{\mu\nu} &= \frac{\partial\partial_\lambda X^\lambda}{\partial(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\partial_\lambda X^\lambda\\ \delta Q &= \int \frac{\partial}{\partial\dot{\phi}}(\partial_\lambda X^\lambda)\dot{\phi} - \partial_\lambda X^\lambda\, d^3x\\ &= -2\int \partial_\lambda X^\lambda\, d^3x\\ &= -2\partial_0\int X^0 d^3x\\ &= 0\end{aligned}$$

where the last equality is due to the fact that X^μ must vanish at the boundary.

For electromagnetism, the energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial^\mu A_\lambda)}\partial_\nu A_\lambda + \frac{1}{4}g_{\mu\nu}F^2 \\ &= -F_{\mu\lambda}\partial_\nu A_\lambda + \frac{1}{4}g_{\mu\nu}F^2 \end{aligned}$$

The first term is generally not symmetry.

Assume $\tilde{T}_{\mu\nu}$ is our new symmetrized tensor, then we must have

$$\tilde{T}_{\{\mu\nu\}} = -F_{\mu\lambda}\partial_\nu A_\lambda + F_{\nu\lambda}\partial_\mu A_\lambda + \frac{\partial\partial_\sigma X^\sigma}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \frac{\partial\partial_\sigma X^\sigma}{\partial(\partial_\nu A_\lambda)}\partial_\mu A_\lambda$$

Thus we must have

$$\frac{\partial\partial_\sigma X^\sigma}{\partial(\partial_\mu A_\lambda)} = F_{\mu\lambda}$$

For which we can choose

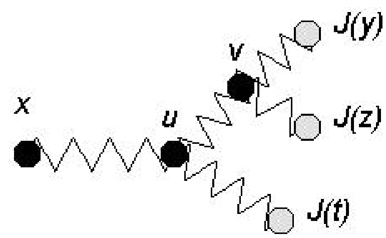
$$X^\sigma = \frac{1}{2}F^{\sigma\tau}A_\tau$$

which gives

$$\begin{aligned} \frac{\partial\partial_\sigma X^\sigma}{\partial(\partial^\mu A_\lambda)} &= \frac{\partial}{\partial(\partial_\mu A_\lambda)}(\frac{1}{2}F^{\sigma\tau}\partial_\sigma A_\tau) \\ &= F_{\mu\lambda} \end{aligned}$$

and thus leads to a symmetric energy-momentum tensor.

Problem 3.4 The Feynman diagram is given by



where we have 5 points x, u, v, y, z, t. The point v can be on either the upper or lower branch, so we have 2 diagrams in total.

Using the Feynman rules given in the chapter, we have

$$\begin{aligned} h_2(x) =& -\lambda^2 \int d^4u\, d^4v\, d^4y\, d^4z\, d^4t\, \Pi(x,u)\Pi(u,t)\Pi(u,v)\Pi(v,y)\Pi(v,z)J(y)J(z)J(t) \\ & -\lambda^2 \int d^4u\, d^4v\, d^4y\, d^4z\, d^4t\, \Pi(x,u)\Pi(u,y)\Pi(u,v)\Pi(v,t)\Pi(v,z)J(y)J(z)J(t) \\ =& -2\lambda^2 \int d^4u\, d^4v\, d^4y\, d^4z\, d^4t\, \Pi(x,u)\Pi(u,t)\Pi(u,v)\Pi(v,y)\Pi(v,z)J(y)J(z)J(t) \end{aligned}$$

From perturbation theory we have

$$\begin{aligned}\Box_u h_2 &= 2\lambda h_1(u)h_0(u)\\ &= 2\lambda^2\int d^4v\, d^4y\, d^4z\, \Pi(u,v)\Pi(v,y)\Pi(v,z)J(y)J(z)\int d^4t\, \Pi(u,t)J(t)\\ h_2 &= -2\lambda^2\int d^4u\, d^4v\, d^4y\, d^4z\, d^4t\, \Pi(x,u)\Pi(u,t)\Pi(u,v)\Pi(v,y)\Pi(v,z)J(y)J(z)J(t)\end{aligned}$$

So the two results match.

Problem 3.5 The equation of motion is

$$\begin{aligned}-\Box\phi + m^2\phi - \frac{\lambda}{3!}\phi^3 &= 0\\ m^2c &= \frac{\lambda}{3!}c^3\\ c &= 0,\, \pm\sqrt{\frac{6}{\lambda}}m\end{aligned}$$

$$V(0)=0,\, V(\pm\sqrt{\frac{6}{\lambda}}m) = -\frac{3}{2}\frac{m^4}{\lambda}$$

The non-zero solutions are the ones corresponding to the ground state.

The only state that respect the $\mathbb{Z}_2$ symmetry is the state $\phi = -\phi = 0$. The ground state picks out either the positive or the negative direction so that symmetry is broken.

If we expand around the ground state, we get

$$\mathcal{L} = -\frac{1}{2}(c+\pi)\Box\pi + \frac{1}{2}m^2c^2 + m^2c\pi + \frac{1}{2}m^2\pi^2 - \frac{\lambda}{4!}(c^4 + 4c^3\pi + 6c^2\pi^2 + 4c\pi^3 + \pi^4)$$

We see that now there are some odd powers of π in the Lagrangian that breaks the $\mathbb{Z}_2$ symmetry. The equation of motion is

$$-\Box\pi + m^2\pi - \frac{\lambda}{2!}c^2\pi - \frac{\lambda}{2!}c\pi^2 - \frac{\lambda}{3!}\pi^3 = 0$$

The $\mathbb{Z}_2$ transformation of ϕ corresponds to $\pi \to -\pi - 2c$. The Lagrangian is obviously invariant in this transformation (because this is just $\phi \to -\phi$!).

Problem 3.6 The equation of motion is given by the Proca equation.

$$\partial_\mu F_{\mu\nu} + m^2 A_\nu = J_\nu$$

Taking divergence we have

$$\begin{aligned}\partial_\mu\partial_\nu F_{\mu\nu} + m^2\partial_\nu A_\nu &= 0\\ \partial_\nu A_\nu &= 0\end{aligned}$$

The last line we use the fact the $F_{\mu\nu}$ is anti-symmetric so the double derivative vanishes.

The equation for A_0 for a point charge is

$$\begin{aligned}
\partial_\mu(\partial_\mu A_0 - \partial_0 A_\mu) + m^2 A_0 &= e\delta^{(3)}(x) \\
(\Box + m^2)A_0 &= e\delta^{(3)}(x) \\
A_0 &= \frac{e}{-\Delta + m^2}\delta^{(3)}(x) \\
A_0 &= \int \frac{d^3k}{(2\pi)^3}\frac{e}{k^2+m^2}e^{ik\cdot x} \\
&= \frac{e}{(2\pi)^2}\int_0^\infty dk \frac{k^2}{k^2+m^2}\frac{e^{ikr}-e^{-ikr}}{ikr} \\
&= \frac{e}{4\pi^2 ir}\Big[\int_0^\infty dk\frac{k}{k^2+m^2}e^{ikr} - \int_0^\infty dk\frac{k}{k^2+m^2}e^{-ikr}\Big] \\
&= \frac{e}{4\pi^2 ir}\int_{-\infty}^\infty dk\frac{k}{k^2+m^2}e^{ikr}
\end{aligned}$$

where in the second line we used the fact that the potential is static so its time derivative vanished.

Evaluating this with contour integration gives

$$\begin{aligned}
A_0 &= \frac{e}{2\pi r}\frac{im}{2im}e^{-mr} \\
&= \frac{e}{4\pi r}e^{-mr}
\end{aligned}$$

In the limit $m \to 0$ (massless photon), we recover the Coulomb potential $A_0 = e/4\pi r$, as expected.

The main difference between the Yukawa potential and the Coulomb potential is that the Yukawa potential has a characteristic range given by $R \sim 1/m$.

For this to be a candidate for force between proton, the characteristic range should be on the order of the proton radius, which corresponds to a mass of $m \sim 1\,\mathrm{fm}^{-1} \approx 200\,\mathrm{MeV}$.

Let us try to substitute the gauge constraint into the Lagrangian.

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu) + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu \\
&= \frac{1}{2}(A_\nu \Box A_\nu - A_\nu \partial_\nu \partial_\mu A_\mu) + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu \\
&= \frac{1}{2}A_\mu(\Box + m^2)A_\mu - A_\mu J_\mu
\end{aligned}$$

where in the second line we have integrated by parts.

The equation of motion becomes

$$\Box A_\mu + m^2 A_\mu = J_\mu$$

The constraint now becomes

$$\begin{aligned}
(\Box + m^2)(\partial_\mu A_\mu) &= 0 \\
0 &= 0
\end{aligned}$$

In the original Lagrangian the mass acts as a Lagrange multiplier. If we turn off the mass the constraint will vanish.

Problem 3.7 The Lagrangian has the unit of energy per volume, i.e. mass power is 4. Thus, looking at the first term, we know that h^2 has unit of energy per length so h has mass power 1. Therefore $a = b = -1$.

The first order solution is just

$$h^{(1)} = -\frac{M_{Pl}^{-1} m}{4\pi r}$$

The appearance of the additional 4π is just because some factor of 4π has been dropped in the Lagrangian. We can drop the factor of 4π and instead write

$$h^{(1)} = -\frac{M_{Pl}^{-1} m}{r}$$

The second-order correction is given by

$$\begin{aligned}\Box h^{(2)} &= M_{Pl}^{-1} \Box (h^{(1)})^2 \\ h^{(2)} &\sim M_{Pl}^{-1} (h^{(1)})^2 \\ &= \frac{M_{Pl}^{-3} m^2}{r^2}\end{aligned}$$

The orbital frequency and the Newtonian potential $u = M_{Pl}^{-1} h$ are related by

$$\begin{aligned}\omega^2 R &= |\frac{\partial u}{\partial r}|_{r=R} \\ \omega^2 &= \frac{G_N m_{Sun}}{R^3} \\ \omega &\sim 10^{-7}\,\mathrm{s}^{-1}\end{aligned}$$

The correction to ω is given by

$$\begin{aligned}\delta\omega &= \frac{1}{2\omega}\delta(\omega^2) \\ &= \frac{1}{\omega}\frac{G_N^2 m_{Sun}^2}{c^2 R^4} \\ &\sim 10^{-14}\,\mathrm{s}^{-1} \\ &\sim 10\,\mathrm{arcsec/century}\end{aligned}$$

Note that factors of c has to be restored as needed to produce the correct dimension.

The effect from other planets can be estimated to be

$$\begin{aligned}\delta\omega &\sim \frac{1}{\omega}\sum \frac{G m_i}{r_i^3} \\ &\sim 10^{-12}\,\mathrm{s} \\ &\sim 10^3\,\mathrm{arcsec/century}\end{aligned}$$

If we derive (3.91) from (3.90), we get an additional term $\sim M_{Pl}^{-1} h \Box h$. This term is the same order as $M_{Pl}^{-1} \Box h^2$. For order-of-magnitude estimates dropping a few terms with the same order doesn't really matter.

Problem 3.9 Let us substitute the gauge condition $\partial_\mu A_\mu$ into the Lagrangian a la problem 3.6 to get

$$\mathcal{L} = \frac{1}{2} A_\mu \Box A_\mu - J_\mu A_\mu$$

The equation of motion is given by

$$\Box A_\mu = J_\mu$$

Substituting this back to the Lagrangian gives

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2} A_\mu \Box A_\mu \\ &= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\, J_\mu(k)\frac{1}{k^2}J_\mu(k')e^{i(k+k')x} \\ &\sim -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, J_\mu(k)\frac{1}{k^2}J_\mu(-k)\end{aligned}$$

Choosing $k_\mu = (\omega, \boldsymbol{k}, 0, 0)$, we have, in momentum space,

$$\begin{aligned}k_\mu J_\mu(k) &= 0 \\ \omega J_0(k) &= \boldsymbol{k} J_1(k)\end{aligned}$$

Therefore we can rewrite the Lagrangian as

$$\begin{aligned}\mathcal{L} &\sim J_\mu(k)\frac{1}{k^2}J_\mu(-k) \\ &= \frac{1}{\omega^2-\boldsymbol{k}^2}\left[J_0(k)J_0(-k) - \frac{\omega^2}{\boldsymbol{k}^2}J_0(k)J_0(-k) - J_2(k)J_2(-k) - J_3(k)J_3(-k)\right] \\ &= -\frac{1}{\boldsymbol{k}^2}J_0(k)J_0(-k) - \frac{1}{\omega^2-\boldsymbol{k}^2}\left[J_2(k)J_2(-k) + J_3(k)J_3(-k)\right]\end{aligned}$$

Thus the time derivative of J_0 disappeared.

$$\begin{aligned}\mathcal{L} &\sim \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\frac{1}{\boldsymbol{k}^2}J_0(k)J_0(k')e^{i(k+k')x} + \cdots \\ &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}J_0(t,\boldsymbol{k})\frac{1}{\boldsymbol{k}^2}J_0(t,\boldsymbol{k}')e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}} + \cdots\end{aligned}$$

If we transform back to position space, we note that the first term couples J_0 at different points in space but at the same moment in time, thus being non-local. However, this non-local degree of freedom is not physical as it can be removed by further gauge fixing. For example choosing the gauge $A_0 = 0$ will remove all appearances of J_0 in the Lagrangian.

Problem 4.1 Both electrons and muons couple to the photon field with the same strength. Thus the interaction can be written as

$$V = \frac{1}{2}e \int d^3x\, \psi_e \phi \psi_e + \psi_\mu \phi \psi_\mu$$

Note that here we have neglected the difference between particle and anti-particle but for a rough argument like this it doesn't really matter.

Again the first order term is zero because both initial and final states contain no photon.

The retarded and advanced intermediate states are $|n^{(R)}\rangle = |\phi^\gamma\rangle$ and $|n^{(A)}\rangle = |\psi_e^1 \psi_e^2 \phi^\gamma \psi_\mu^3 \psi_\mu^4\rangle$. The relevant matrix elements are

$$\begin{aligned}
V_{ni}^{(R)} &= \frac{1}{2}e\langle\phi^\gamma| \int d^3x\, \psi_e \phi \psi_e |\psi_1^e \psi_2^e\rangle \\
&= e(2\pi)^3 \delta^{(3)}(p_1 + p_2 - p_\gamma) \\
V_{fn}^{(R)} &= \frac{1}{2}e\langle\psi_3^\mu \psi_4^\mu| \int d^3x\, \psi_\mu \phi \psi_\mu |\phi^\gamma\rangle \\
&= e(2\pi)^3 \delta^{(3)}(p_3 + p_4 - p_\gamma) \\
V_{ni}^{(A)} &= \langle\psi_e^1 \psi_e^2 \phi^\gamma \psi_\mu^3 \psi_\mu^{\dagger 4}|V|\psi_e^1 \psi_e^2\rangle \\
&= \langle\phi^\gamma \psi_\mu^3 \psi_\mu^{\dagger 4}|V|0\rangle\langle\psi_e^1 \psi_e^{\dagger 2}|\psi_e^1 \psi_e^{\dagger 2}\rangle \\
&= e(2\pi)^3 \delta^{(3)}(p_3 + p_4 + p_\gamma) \\
V_{fn}^{(A)} &= e(2\pi)^3 \delta^{(3)}(p_1 + p_2 + p_\gamma)
\end{aligned}$$

Therefore the transfer matrix element is

$$\begin{aligned}
T_{fi}^{(R)} &= e^2 \int dp_\gamma\, (2\pi)^6 \Big[\frac{\delta^{(3)}(p_3 + p_4 - p_\gamma)\delta^{(3)}(p_1 + p_2 - p_\gamma)}{E_i - E_n}\Big] \\
&= e^2 \frac{1}{E_1 + E_2 - E_\gamma} \\
&= e^2 \frac{1}{E' - E_\gamma}
\end{aligned}$$

$$\begin{aligned}
T_{fi}^{(A)} &= e^2 \int dp_\gamma\, (2\pi)^6 \Big[\frac{\delta^{(3)}(p_3 + p_4 + p_\gamma)\delta^{(3)}(p_1 + p_2 + p_\gamma)}{E_i - E_n}\Big] \\
&= e^2 \frac{1}{E_1 + E_2 - (E_1 + E_2 + E_\gamma + E_3 + E_4)} \\
&= e^2 \frac{1}{-(E_3 + E_4) - E_\gamma} \\
&= e^2 \frac{1}{-E' - E_\gamma}
\end{aligned}$$

where $E' = E_1 + E_2 = E_3 + E_4$. If we let $k = (E', p_\gamma)$ be the virtual off-shell momentum of the photon, then the transfer matrix is simply

$$
\begin{aligned}
T_{fi} &= T_{fi}^{(R)} + T_{fi}^{(A)} \\
&= e^2 \frac{2E_\gamma}{E'^2 - E_\gamma^2} \\
&= 2E_\gamma \left(\frac{e^2}{k^2} \right)
\end{aligned}
$$

Problem 5.1

$$\begin{aligned} d\Pi_{LIPS} &= (2\pi)^4\delta^4(\sum p)\frac{d^3p_f}{(2\pi)^3}\frac{d^3p_B}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{2E_B} \\ &= \frac{d\Omega}{16\pi^2}\int \delta(E_f+E_B-E_i-m_A)p_f^2dp_f\frac{1}{E_f}\frac{1}{E_B} \end{aligned}$$

To proceed we change variable from p_f to $x(p_f) = E_f + E_B - E_i - m_A$ using

$$\begin{aligned} \frac{dx}{dp_f} &= \frac{dE_f}{dp_f} + \frac{dE_B}{dp_f} = \frac{p_f}{E_f} + \frac{p_f + p_i\cos\theta}{E_B} \\ d\Pi_{LIPS} &= \frac{d\Omega}{16\pi^2}\Big[p_f^2\frac{1}{E_Bp_f + E_fp_f + E_fp_i\cos\theta}\Big] \\ &= \frac{d\Omega}{16\pi^2}\Big[\frac{1}{E_B + E_f(1+\frac{p_i}{p_f}\cos\theta)}\Big]p_f \\ \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 m_A}\Big[\frac{1}{E_B + E_f(1+\frac{p_i}{p_f}\cos\theta)}\Big]\frac{p_f}{p_i}|\mathcal{M}|^2 \end{aligned}$$

where we have used $v_i = p_i/E_i$.

Problem 5.2 See problem 2.6

Problem 5.3 Let us work in the rest frame of the decaying muon and take the direction of the outgoing electron neutrino to be the z-axis.

$$\begin{aligned} d\Pi_{LIPS} &= (2\pi)^4\delta^4(\sum p)\frac{d^3p_e}{(2\pi)^3}\frac{d^3p_{\nu e}}{(2\pi)^3}\frac{d^3p_{\nu\mu}}{(2\pi)^3}\frac{1}{2E_e}\frac{1}{2E_{\nu e}}\frac{1}{2E_{\nu\mu}} \\ &= \frac{1}{8(2\pi)^5}\int\frac{4\pi p_e dp_e E dE d\Omega}{\sqrt{p_e^2+E^2+2p_eE\cos\theta}} \\ &= \frac{1}{4(2\pi)^3}\int\delta(m-p_e-E-p_{\nu\mu})p_edp_eEdE\int_{-1}^{1}\frac{d\cos\theta}{\sqrt{p_e^2+E^2+2p_eE\cos\theta}} \\ &= \frac{1}{4(2\pi)^3}\int\delta(m-p_e-E-p_{\nu\mu})dp_eEdE \\ &= \frac{1}{4(2\pi)^3}EdE \\ \Gamma &= \frac{32G_F^2}{8(2\pi)^3}\int_0^{m/2}(m^2-2mE)E^2dE \\ &= \frac{G_F^2m^5}{192\pi^3} \end{aligned}$$

Note that the upper bound for E is $m/2$ because otherwise there is no way to balance the momentum such that sum of momentum is zero.

If we put the muon mass $m = 106\,\text{MeV}$, we get

$$\begin{aligned}\Gamma &= 3\times 10^{-16}\,\text{MeV}\\ \tau = \Gamma^{-1} &= 6.5\times 10^{17}\,\text{fm}\\ &= 2.18\,\mu\text{s}\end{aligned}$$

The percentage discrepancy is around 1%, which might be due to non-zero electron mass ($m_e \sim 0.5\,\text{MeV} \sim 1\%\,m_\mu$).

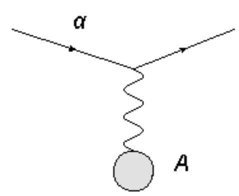
Problem 5.5 The classical Rutherford cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4E_K^2\sin^4(\theta/2)}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant. This assumes that the alpha particle is non-relativistic and the nucleus is heavy enough that we can neglect the recoil.

We replace e^2 by $2Ze^2$ and m_e by $m_\alpha$to get

$$\frac{d\sigma}{d\Omega} = \frac{16Z^2\alpha^2 m_\alpha^2}{k^4}$$



The momentum of the virtual photon is just equal to the change of momentum in the alpha particle which is given by

$$\begin{aligned}k_\mu &= m_\alpha v(0, \cos\theta - 1, \sin\theta, 0)\\ k^2 &= 2m_\alpha^2 v^2(1-\cos\theta)\\ &= 8m_\alpha E_K \sin^2(\theta/2)\end{aligned}$$

Therefore

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4E_K^2\sin^4(\theta/2)}$$

which matches with the classical expression.

There is no real reason why the leading contribution from both classical and quantum mechanics should match so there is no way to know ahead of time. Rutherford was very lucky in this regard.

For electron-electron (Møller) scattering this formula breaks down because we can no longer neglect the recoil and treat the other electron as a static field. Also there are important contributions coming from the spin structure of the electrons.

Problem 6.1 We first calculate the case for $\Delta t = 0$ and promote our answer using Lorentz invariance.

$$\begin{aligned}
D_F(x_1 - x_2) &= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{ik(x_1 - x_2)} \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{dk^0}{2\pi} \frac{i}{(k^0)^2 - \omega^2 + i\epsilon} e^{ik^0(t_1 - t_2)} e^{-ik\cdot(x_1 - x_2)} \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega + i\epsilon} e^{-i\omega|t_1 - t_2|} e^{-ik\cdot(x_1 - x_2)} \\
&= i \int \frac{dk}{(2\pi)^2} \frac{k}{2r\omega + i\epsilon} e^{-i\omega|\Delta t|} (e^{-ikr} - e^{ikr}) \\
&= \int \frac{dk}{(2\pi)^2} \frac{k}{r\sqrt{k^2 + m^2} + i\epsilon} \sin kr \\
&= \frac{m}{4\pi^2 r + i\epsilon} K_1(mr) \\
&\to \frac{1}{4\pi^2 r^2 + i\epsilon}
\end{aligned}$$

Therefore by Lorentz symmetry the propagator must be

$$D_F = -\frac{1}{4\pi^2(\Delta t^2 - r^2) + i\epsilon} = -\frac{1}{4\pi^2(x_1 - x_2)^2 + i\epsilon}$$

Problem 6.2

$$\begin{aligned}
D_R &= \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} (e^{-ik(x-y)} - e^{ik(x-y)}) \\
&= \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{ik\cdot(x-y)} (e^{-i\omega_k(x^0 - y^0)} - e^{i\omega_k(x^0 - y^0)}) \\
&= \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x-y)} \int \frac{dk^0}{2\pi i} \frac{-1}{(k^0 + i\epsilon)^2 - \omega_k^2} e^{-ik^0(x^0 - y^0)} \\
&= i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{(k^0 + i\epsilon)^2 - k^2 - m^2}
\end{aligned}$$

where we have use the mathematical identity

$$\begin{aligned}
\int \frac{dk^0}{2\pi i} \frac{-1}{(k^0 + i\epsilon)^2 - \omega_k^2} e^{-ik^0(x^0 - y^0)} &= \begin{cases} \frac{1}{2\omega_k} (e^{-i\omega_k(x^0 - y^0)} - e^{i\omega_k(x^0 - y^0)}) & x^0 - y^0 > 0 \\ 0 & x^0 - y^0 \leq 0 \end{cases} \\
&= \theta(x^0 - y^0) \frac{1}{2\omega_k} (e^{-i\omega_k(x^0 - y^0)} - e^{i\omega_k(x^0 - y^0)})
\end{aligned}$$

Similarly we have

$$\begin{aligned} D_A &= \theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} (e^{-ik(x-y)} - e^{ik(x-y)}) \\ &= \theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{ik\cdot(x-y)} (e^{i\omega_k(y^0-x^0)} - e^{-i\omega_k(y^0-x^0)}) \\ &= \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x-y)} \int \frac{dk^0}{2\pi i} \frac{1}{(k^0 - i\epsilon)^2 - \omega_k^2} e^{ik^0(y^0-x^0)} \\ &= -i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{(k^0 - i\epsilon)^2 - k^2 - m^2} \end{aligned}$$

Problem 6.3 The operator $\mathcal{O}$ can be written in terms of its matrix elements:

$$\mathcal{O} = \sum_{nm} \int d^n q d^m p \, |q_1 q_2 ... q_n\rangle O_{nm}(q_1, ..., q_n; p_1, ..., p_m) \langle p_1 p_2 ... p_m|$$

We claim this is equivalent to

$$\mathcal{O} = \sum_{nm} \int d^n q d^m p \, a^\dagger_{q_1} ... a^\dagger_{q_n} a_{p_1} ... a_{p_m} C_{nm}(q_1, ..., q_n; p_1, ..., p_m)$$

Note that the ordering of q's and p's does not matter because all the creation operators commute with each other and all the annihilation operators commute with each other. Thus both O_{nm} and C_{nm} have the same permutation symmetry in their arguments[1]. This is important for the proof to work.

We have to show that we can reproduce all the matrix elements correctly. First it is obvious that we can reproduce the vacuum expectation by choosing $C_{00} = O_{00}$. Now assume we have chosen C_{nm} such that all matrix elements of order up to $n = N$ and $m = M$ are correctly reproduced. Let us consider the case when $n = N + 1$, $m = M$. For simplicity we have dropped normalization factors which does not affect the result.

$$O_{(N+1)M}(q, q_{N+1}; p) = \langle 0| a_{q_1} ... a_{q_{N+1}} \sum_{nm} \int d^n q' d^m p' \, a^\dagger_{q'_1} ... a^\dagger_{q'_n} a_{p'_1} ... a_{p'_m} C_{nm}(q'_1, ..., q'_n; p'_1, ..., p'_m) \, |p\rangle$$

On the right hand side there will be two type of terms. The first type involves the commutators $[a_{q_{N+1}}, a^\dagger_{q'_i}]$ which gives terms involving $C_{(N+1)M}$. There are $(N+1)!M!$ such terms due to the different combinations and although the different combinations result in different ordering among the q's and the p's, the value of $C_{(N+1)M}$ for all these permutations of the momenta is the same. The second group of terms involves commutators $[a_{q_{N+1}}, a^\dagger_{p_i}]$ between operators in the left and right states. The remaining operators will pair up resulting in terms including C_{nm} of order $n \le N$, $m \le M$ which have been already fixed due to our assumption. Therefore we have

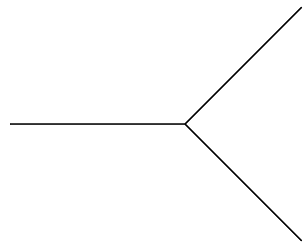
$$O_{(N+1)M}(q; p) = (N+1)!M! C_{(N+1)M}(q; p) + (\text{fixed terms})$$

which gives us the correct choice for $C_{(N+1)M}(q; p)$. The case where $n = N$ and $m = M + 1$ is almost identical. Therefore by induction the claim is true.

[1] The proof also works for anti-commutating operators thanks to this because O_{nm} and C_{nm} will pick up the same minus sign which then will cancel out.

Problem 7.1

a



$$\mathcal{M} = g$$

b

$$i\mathcal{M} \sim (ig)^3 \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{k_3^2 - m^2 + i\varepsilon}$$

We need to conserve momentum. Take clockwise as the positive direction for the internal momenta k_1, k_2, k_3, we have

$$p_1 + k_3 = k_1$$
$$p_2 + k_2 = k_1$$
$$p_3 + k_3 = k_2$$

Thus, let $k_3 = k$, we have

$$i\mathcal{M} = -g^3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p_1 + k)^2 - m^2 + i\varepsilon} \frac{1}{(p_3 + k)^2 - m^2 + i\varepsilon} \frac{1}{k^2 - m^2 + i\varepsilon}$$

c In position space we have

$$\langle \phi(a)\phi(b)\phi(c) \rangle = -ig^3 \int d^4x\, d^4y\, d^4z\, D_F(a,x) D_F(x,y) D_F(y,b) D_F(y,z) D_F(z,c) D_F(z,x)$$

d By the LSZ formula (we suppress $i\varepsilon$ for clarity)

$$
\begin{aligned}
i\mathcal{M} &= \Big[i\int d^4a\, e^{-ip_1 a}(\Box_a + m^2)\Big]\Big[...\Big]_b\Big[...\Big]_c\Big(-ig^3\int d^4x\, d^4y\, d^4z\, D_F(a,x)D_F(x,y)D_F(y,b)D_F(y,z)D_F(z,c)D_F(z,x)\Big)\\
&= -ig^3\int d^4x\, d^4x\, d^4y\, d^4z\, e^{ip_1 x}D_F(x,y)e^{ip_2 y}D_F(y,z)e^{ip_3 z}D_F(z,x)\\
&= -g^3\int d^4x ... \frac{d^4k_1}{(2\pi)^4}...\frac{1}{k_1^2-m^2}\frac{1}{k_2^2-m^2}\frac{1}{k_3^2-m^2}e^{ip_1x}e^{ik_1(x-y)}e^{ip_2y}e^{ik_2(y-z)}e^{ip_3z}e^{ik_3(z-x)}\\
&= -g^3\int d^4k_1...\,\delta^{(4)}(p_1+k_1-k_3)\delta^{(4)}(p_2-k_1+k_2\}\delta^{(4)}(p_3+k_3-k_2)\frac{1}{k_1^2-m^2}\frac{1}{k_2^2-m^2}\frac{1}{k_3^2-m^2}\\
&= -g^3\delta^{(4)}(p_1-p_2-p_3)\int d^4k\frac{1}{(p_1-k)^2-m^2}\frac{1}{(p_3-k)-m^2}\frac{1}{k^2-m^2}\\
&= -g^3\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p_1+k)^2-m^2}\frac{1}{(p_3+k)-m^2}\frac{1}{k^2-m^2}
\end{aligned}
$$

which is exactly the same as the expression in (b). In the last line we dropped the delta function of total momentum.

Problem 7.2 Using the LSZ formula gives us

$$\mathcal{M}_{6pt} = \lambda$$

$$\delta^{(4)}(\sum p)\mathcal{M}_{3pt} = g^2\delta^{(4)}(p_1-p_3-p_4)\delta^{(4)}(p_2-p_5-p_6) + (\text{permutations})$$

The 3-point amplitude has an additional delta function, as pointed out in section 7.3.2.

Problem 7.3

a We have the usual s,t and u channel scatterings.

b The s channel is forbidden due to charge conservation.

c

$$i\mathcal{M} = \frac{im_e^2e^2}{t} - \frac{im_e^2e^2}{u}$$

d The connected pair of electrons must have the same spin. Therefore there are $2\times 2 = 4$ combinations for each of t and u processes. Out of these processes only 2 are common to both and hence can interfere. The remaining 8 processes with unequal number of up/down spins are forbidden.

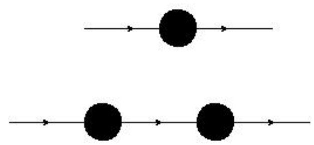
e

$$|\mathcal{M}|^2 = m_e^4 e^4 \Big[\frac{1}{u^2} + \frac{1}{t^2} - \frac{1}{2}\Big(\frac{2}{ut}\Big)\Big]$$
$$= \frac{m_e^4 e^4}{4p^4}\Big[\frac{1}{(1-\cos\theta)^2} + \frac{1}{(1+\cos\theta)^2} - \frac{1}{1-\cos^2\theta}\Big]$$

(cross-section?)

Problem 7.4

a Neglecting the disconnected graphs, we have



etc.
The black dot denotes mass coupling m^2.

b

$$G_0 = D_F(x,y)$$
$$= \int d^4k\, e^{ik(x-y)} \frac{i}{k^2}$$
$$G_1 = -im^2 \int d^4u\, D_F(x,u) D_F(u,y)$$
$$= -im^2 \int d^4k\, e^{ik(x-y)} \Big(\frac{i}{k^2}\Big)^2$$
$$G_2 = (-im^2)^2 \int d^4u\, d^4v\, D_F(x,u) D_F(u,v) D_F(v,y)$$
$$= (-im^2)^2 \int d^4k\, e^{ik(x-y)} \Big(\frac{i}{k^2}\Big)^3$$

etc.

c Summing over the series gives

$$G = \int d^4k\, \frac{i}{k^2} \sum_n \Big(- im^2 \frac{i}{k^2}\Big)^n$$
$$= \int d^4k\, \frac{i}{k^2} \frac{1}{1 - \frac{m^2}{k^2}}$$
$$= \int d^4k\, \frac{i}{k^2 - m^2}$$

which is just the massive propagator.

d The equation for the propagator is

$$\Box G + m^2 G = -i\delta^{(4)}(x-y)$$

The zeroth order solution is

$$G^{(0)} = \int d^4k\, e^{ik(x-y)}\frac{i}{k^2}$$

The first order solution is

$$\begin{aligned}\Box G^{(1)} + m^2 G^{(0)} &= 0\\ G^{(1)} &= \frac{-m^2}{\Box}G^{(0)}\\ G^{(1)} &= \int d^4k\, e^{ik(x-y)}\frac{i}{k^2}\frac{m^2}{k^2}\end{aligned}$$

All higher orders are given by the recursion

$$G^{(n+1)} = \frac{-m^2}{\Box}G^{(n)}$$

which leads to

$$G^{(n)} = \int d^4k\left(\frac{m^2}{k^2}\right)^n\frac{i}{k^2}e^{ik(x-y)}$$

which again sums to the massive propagator.

Problem 7.5

Problem 7.8

a

$$\mathcal{L} = -\frac{1}{2}e\Box e - \frac{1}{2}\mu(\Box + m_\mu^2)\mu - \frac{1}{2}\nu_\mu\Box\nu_\mu - \frac{1}{2}\nu_e\Box\nu_e - \frac{1}{2}W(\Box + m_W^2)W + g(\mu\nu_\mu W + e\nu_e W)$$

b We work in the center of momentum frame and take the direction of the outgoing muon neutrino as the z-axis.

$$\begin{aligned}i\mathcal{M} &= (ig)^2\frac{i}{(p_\mu - p_{\nu\mu})^2 - m_W^2}\\ |\mathcal{M}|^2 &= \frac{g^4}{m_W^4}\left(1 + \frac{2(m_\mu^2 - m_\mu E_{\nu\mu})}{m_W^2}\right)\end{aligned}$$

c $\mathcal{M}$ is dimensionless so g appears to have dimension of mass. We can make it dimensionless by inserting m_μ.

$$|\mathcal{M}|^2 = \frac{g^4 m_\mu^4}{m_W^4}\Big(1 + \frac{2(m_\mu^2 - m_\mu E_{\nu\mu})}{m_W^2}\Big)$$

The decay rate is (after integrating over $E_{\nu\mu}$)

$$\Gamma = \frac{g^4 m_\mu^5}{192\pi^3 m_W^4}\Big(\frac{1}{2} + \frac{3m_\mu^2}{4m_W^2}\Big)$$

d To give an order of magnitude estimate of m_W, let us take $g \sim 0.1$. Ignoring the second term in the bracket we can estimate m_W to be

$$\begin{aligned}\Big(\frac{m_W}{g}\Big)^4 &\approx \frac{m_\mu^5}{192\pi^3}\frac{T}{2} \\ m_W &\approx 250g\,\text{GeV} \\ m_W &\sim O(10)\,\text{GeV}\end{aligned}$$

e We take the ratio of lifetimes to have

$$\begin{aligned}\Big(\frac{m_\tau}{m_\mu}\Big)^5 &= \frac{T_\mu}{T_\tau} \\ m_\tau &\approx 2500\,\text{MeV}\end{aligned}$$

f

$$\begin{aligned}\Gamma &= 17.8\%\Gamma_{tot} \\ m_\tau &\approx 1767\,\text{MeV}\end{aligned}$$

g We can measure the decay width of muon up to leading order to get the ratio g/m_W and then measure the decay width of tau up to NLO to fit out m_W and thus g. Of course we can also measure both widths up to NLO, which would require much higher precision. Assuming we know the mass of muon and tau to high precision, either of these will allow us to fit both g and m_W , rather than just the ratio. The minimum precision we need is on the order of

$$\frac{\Delta T}{T} \sim \Big(\frac{m_\tau}{m_W}\Big)^2 \sim 0.05\%$$

Problem 7.7

a Due to the external legs the symmetry factor is 1. To write down the amplitude using Feynman rules we first need to fully implement all the momentum conservations at vertices. Denoting the external momenta as p_i's (all outgoing except p_1) and the internal momenta as k_i's, we have

$$\begin{aligned} p_1 &= k_1 + k_2 + k_3 \\ p_2 &= k_1 - k_4 - k_5 \\ p_3 &= k_3 + k_5 + k_6 \\ p_4 &= k_2 + k_4 - k_6 \end{aligned}$$

This is 4 equations for 6 unknowns but only 3 equations are linearly independent due to total momentum conservation. Writing everything in terms of k_1, k_2, k_3 and p_i's, we have

$$\begin{aligned} k_4 &= \frac{1}{2}(p_1 - k_1 - k_2 - k_3) \\ k_5 &= \frac{1}{2}(p_3 + p_4 - p_2 + k_1 - k_2 - k_3) \\ k_6 &= \frac{1}{2}(p_3 - p_4 + p_2 - k_1 + k_2 - k_3) \end{aligned}$$

Integrating over the undetermined momenta, we have

$$i\mathcal{M} = -\lambda^4 \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}} \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{k_3^2} \frac{1}{k_4^2} \frac{1}{k_5^2} \frac{1}{k_6^2}$$

with k_4, k_5, k_6 given above.

b The symmetry factor simply corresponds to permuatation of the vertices, i.e. $S = 4!$. Thus,

$$i\mathcal{M} = -\frac{\lambda^4}{4!} \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}} \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{k_3^2} \frac{1}{k_4^2} \frac{1}{k_5^2} \frac{1}{k_6^2}$$

with all p_i's set to zero.

Problem 7.9

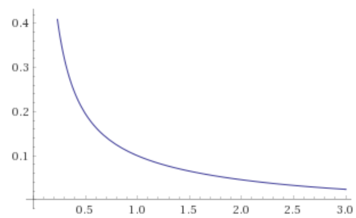
a For s-channel the mediating particle has momentum $s = (p_1 + p_2)^2$ thus

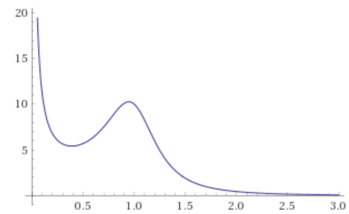
$$\begin{aligned} |\mathcal{M}|^2 &\sim |\frac{1}{s - m^2 + im\Gamma}|^2 \\ \sigma &\sim \frac{1}{s} \frac{1}{(s-m^2)^2 + m^2\Gamma^2} \end{aligned}$$

b Let us take the constant of proportionally to be m^6 by changing to suitable units, and let $x = s/m^2$

$$\sigma(x) = \frac{1}{x} \frac{1}{(x-1)^2 + \left(\frac{\Gamma}{m}\right)^2}$$

For Γ/m large the cross-section looks $1/x$ for small x and decays as x^{-3} for large x (left below). For Γ/m small the cross-section has a sharp peak around $x = 1$ (right below).





c

$$
\begin{aligned}
\mathcal{M} &= \frac{1}{p^2 - m^2 + i\varepsilon} \\
\text{Im}(\mathcal{M}) &= -\lim_{\varepsilon \to 0} \frac{\varepsilon}{(p^2 - m^2)^2 + \varepsilon^2} \\
&= -\pi\delta(p^2 - m^2)
\end{aligned}
$$

where we have use the limit representation of delta function given by $\lim_{x \to 0} x/(x^2 + t^2) = \pi\delta(t)$. Therefore when the particle is off-shell the imaginary part is zero.

d Assume that the amplitude of the loop diagram is $\mathcal{M}_{loop} = A + i\Sigma$. Then we have

$$
\begin{aligned}
D_F^{(dressed)} &= D_F + D_F(i\mathcal{M}_{loop})D_F + D_F(i\mathcal{M}_{loop})D_F(i\mathcal{M}_{loop})D_F + ... \\
&= \frac{i}{p^2 - m_0^2} \frac{1}{1 + \frac{A + i\Sigma}{p^2 - m^2}} \\
&= \frac{i}{p^2 - (m_0^2 - A) + i\Sigma} \\
&= \frac{i}{p^2 - m^2 + i\Sigma}
\end{aligned}
$$

The real part of the loop diagram leads to some renormalization of the mass which we are not interested in. The imaginary part leads to a decay width given by

$$
\Gamma = \Sigma/m
$$

e We can interpret the results of parts c and d using virtual particles. When a particle is propagating it is constantly turning into some virtual particles and back due to interaction with the non-trivial vacuum. For example in part d ϕ is constantly turning into ψ's and back to ϕ. When the kinematics allows, the virtual particles can go on-shell and materialize. Physically then one observes a decay from the original particle to these newly materialized particles. Mathematically, as we have seen in part c, the "on-shelling" leads to a non-zero imaginary part in the amplitude. This in turn lead to a decay width in the propagator as shown in part d. We also see in part d that we expect the decay width to be proportional to the strength of interaction. This makes sense because the stronger the interaction with the virtual sea the higher the chance that the original particle will turn into the decay product.